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VOLUME I

SUPERSONIC FLOW AND SHOCK WAVES

R. COURANT and K. O. FRIEDRICHS
INSTITUTE FOR MATHEMATICS AND MECHANICS
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To

Warren Weaver

Preface

The present book originates from a report issued in 1944 under the auspices of the Office of Scientific Research and Development. Much material has been added and the original text has been almost entirely rewritten. The book treats basic aspects of the dynamics of compressible fluids in mathematical form; it attempts to present a systematic theory of nonlinear wave propagation, particularly in relation to gas dynamics. Written in the form of an advanced textbook, it accounts for classical as well as some recent developments, and, as the authors hope, it reflects some progress in the scientific penetration of the subject matter. On the other hand, no attempt has been made to cover the whole field of nonlinear wave propagation or to provide summaries of results which could be used as recipes for attacking specific engineering problems.

The book has been written by mathematicians seeking to understand in a rational way a fascinating field of physical reality, and willing to accept compromise with empirical approach. The authors hope that it will be helpful to engineers, physicists, and mathematicians alike, and that it will not be rejected by mathematicians as too heavily loaded with physical assumptions or by others as too strictly mathematical.

Dynamics of compressible fluids, like other subjects in which the nonlinear character of the basic equations plays a decisive role, is far from the perfection envisaged by Laplace as the goal of a mathematical theory. Classical mechanics and mathematical physics predict phenomena on the basis of general differential equations and specific boundary and initial conditions. In contrast, the subject of this book largely defies such claims. Important branches of gas dynamics still center around special types of problems, and general features of connected theory are not always clearly discernible. Nevertheless, the authors have attempted to develop and to emphasize as much as possible such general viewpoints, and they hope that this effort will stimulate further advances in this direction.

In a field which during recent years has attracted so many workers and in which such diverse practical and theoretical interests have asserted themselves, the authors found a balanced survey impossible; instead they have followed a path dictated largely by their personal interests and experience. The names of scientists with whom the authors happened to be in close contact appear frequently; names of others may have been omitted. No fair appraisal could be made of the merits of many recent contributions. This is true in particular of the large number of reports issued during the war by various agencies and still not freely accessible. In order to avoid further delay, the authors are publishing this book without a complete survey of the literature.

The book was prepared for publication with the cooperation of members of the staff of the Institute for Mathematics and Mechanics in New York University. The main burden of the editorial work, done for the original report by R. Shaw, has been carried by Cathleen Synge Morawetz, who has also contributed constructive criticism in many details, and whose understanding and competent assistance have been invaluable. L. J. Savage cooperated actively in rewriting the first chapter and other parts of the original report. D. A. Flanders has helped greatly by reading parts of the manuscript and suggesting important improvements. W. Y. Chen, W. M. Hirsch, E. Isaacson, A. Leitner, S. C. Lowell, and M. Sion have assisted in this publication by reading proofs and making useful suggestions. The drawings, many of which represent actual conditions, have been carried out by G. W. Evans and J. R. Knudsen. The preparation of the manuscript was in the competent hands of Edythe Rodermund and Harriet Schoverling.

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Thanks also should be expressed to Interscience Publishers for the cooperative attitude of their staff, and for the genuine interest of their officers in the promotion of scientific publications.

The book is dedicated to Warren Weaver. As chief of the Applied Mathematics Panel during the war, he rendered very great services, not only for the problems of the day, but even more so for the lasting

benefit of the mathematical sciences. For us personally his steady interest in the present work has been a source of encouragement. Thus the dedication of the book is as well a token of friendship as a tribute to a man whose energy and vision have contributed so much to the recent development of applied mathematics in this country.

R. COURANT and K. O. FRIEDRICHS

August, 1948

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CHAPTER I

Compressible Fluids

Violent disturbances—such as result from detonation of explosives, from the flow through rocket nozzles, from supersonic flight of projectiles, or from impact on solids—differ greatly from the "linear" phenomena of sound, light, or electromagnetic signals. In contrast to the latter, their propagation is governed by nonlinear differential equations, and as a consequence the familiar laws of superposition, reflection, and refraction cease to be valid; but even more novel features appear, among which the occurrence of *shock fronts* is the most conspicuous. Across shock fronts the medium undergoes sudden and often considerable changes in velocity, pressure, and temperature. Even when the start of the motion is perfectly continuous, shock discontinuities may later arise automatically. Yet, under other conditions, just the opposite may happen; initial discontinuities may be smoothed out immediately. Both these possibilities are essentially connected with the nonlinearity of the underlying equations.

Nature confronts the observer with a wealth of nonlinear wave phenomena, not only in the flow of compressible fluids, but also in many other cases of practical interest. One example, rather different from those mentioned above, is the catastrophic pressure in a crowd of panicky people who rush toward a narrow exit or other obstruction. If they move at a speed exceeding that at which warnings are passed backward, a pressure wave arises much like that behind a shock front receding from a wall. Related phenomena, such as congestion in traffic, seem to be essentially due to similar conditions. In this book, however, we shall concentrate primarily on the theory of compressible fluids.

Understanding and control of nonlinear wave motion is a matter of obvious importance. During a period beginning almost a hundred years ago, Stokes, Earnshaw, Riemann, Rankine, Hugoniot, Lord

Rayleigh, and later Hadamard and others wrote fundamental papers inaugurating this field of research. Then the development was left mainly to a small group of ingenious men in the fields of mechanics and engineering. During the last few years, however, when the barriers between applied and pure science were forced down, a widespread interest arose in nonlinear wave motion, particularly in shock waves and expansion waves.

It is the purpose of the present book to make the mathematical theory of nonlinear waves more accessible, giving particular attention to some recent developments.*

1. *Qualitative differences between linear and nonlinear waves*

Some characteristics of nonlinear wave motion can be described in general terms. In linear wave motion, as, for example, in the transmission of sound, disturbances are always propagated with a definite speed (relative to the medium) which may vary within the medium. This "sound speed" is a local property of the medium itself and remains the same for every conceivable linear wave motion in the medium. Such a sound speed also plays a role in nonlinear wave motion. Small disturbances or "wavelets," slightly modifying a given primary wave motion, are propagated with a certain speed, again called sound speed, though in this case the sound speed depends not only on the position within the medium but on the state of the medium induced by the primary motion.

The distinctive feature of nonlinear waves, however, concerns disturbances or discontinuities which are not necessarily small. In linear wave motion any initial discontinuity across a surface is preserved as a discontinuity and propagated with sound speed. Nonlinear wave motion behaves in a different manner: Suppose there is an initial discontinuity between two regions of different pressures, densities, and flow velocities. Then there are the following *alternative* possibilities: either the initial discontinuity is resolved immediately and the disturbance, while propagated, becomes continuous, or the initial discontinuity is propagated through one or two *shock fronts*, advancing not at sonic but at supersonic speed relative to the medium

* For the theory of compressible flow reference may be made to [3,4,5]; different approaches are given by Sauer [6] and Liepmann and Puckett [7].

ahead of them. As previously stated, shock fronts are the most conspicuous phenomena occurring in nonlinear wave propagation; even without being caused by initial discontinuities they may appear and be propagated. The underlying mathematical fact is that, unlike linear partial differential equations, nonlinear equations often do not admit solutions which can be continuously extended wherever the differential equations themselves remain regular.

Another striking difference between linear and nonlinear waves concerns the phenomenon of interaction: the principle of superposition holds for linear waves but not for nonlinear waves. As a consequence, for example, excess pressures of interfering sound waves are merely additive; in contrast to this fact, interaction and reflection of nonlinear waves may lead to enormous increases in pressure.

A. General Equations of Flow. Thermodynamic Notions

2. The medium

We shall be primarily concerned with a moving fluid, though many of the results apply to other moving media (e.g. to a solid slab in longitudinal wave motion). In this section we shall set forth the properties of the medium that will be assumed throughout the book and we shall describe certain idealized media of special interest. Moreover, since gas dynamics is thoroughly interwoven with thermodynamical concepts, it is appropriate to insert here a collection of basic notions of thermodynamics in a suitable mathematical form.*

Except where the motion is *discontinuous*, viscosity, heat conduction, and deviation of the medium from thermodynamic equilibrium (at any instant and any point) will be neglected. Some critical comments concerning the neglect of these phenomena will be made in later chapters. In particular it will be shown that viscosity and heat conduction play an important role in forming and maintaining shock discontinuities.

At each instant and each point of the fluid there is a definite state (of thermodynamic equilibrium) defined by:

p the pressure,
T the temperature,

*For textbooks on thermodynamics see Epstein [20] and Zemansky [21].

τ the specific volume (i.e. volume per unit mass),
ρ the density, with $\rho\tau = 1$,
S the specific entropy,
e the specific (internal) energy, and
i the specific enthalpy,* defined by $i = e + p\tau$.

It is known from thermodynamics that for any given medium only two of the parameters p, T, τ, e, and S are independent. In fact they may all be considered as functions of τ and S.

The internal energy gained by the medium during a change from one state to another is the heat contributed to the medium plus the work done on the medium by compressive action of the pressure forces. For a change from one state to an immediately neighboring one this fundamental fact is expressed by the relation

$$de = TdS - pd\tau. \tag{2.01}$$

In a reversible process, TdS is the heat acquired by conduction; in an irreversible process, TdS is greater than the heat so acquired. If the irreversible process is one that can be described as determined by the action of viscosity, then the excess of TdS over the heat acquired by conduction may conveniently be interpreted as the heat produced by viscous forces.

Suppose that for some medium we know how the specific energy e depends on τ and S. Then the pressure p and temperature T may immediately be found on considering the meaning of relation (2.01). Thus

$$p = -e_\tau\,, \; T = e_S\,, \tag{2.02}$$

the subscripts indicating partial differentiation.**

The functions giving p in terms of ρ, or τ, and S, occurring so frequently in the theory of fluid flow, will consistently be denoted by

$$p = f(\rho, S)\,; \; p = g(\tau, S). \tag{2.03}$$

Extending slightly the conventional nomenclature, we shall call either of these equations the *caloric equation of state* of the medium.

Neglecting viscosity and heat conduction is tantamount to assuming that as a particle of the medium moves about, the specific

* The notion of enthalpy will be discussed in Section 9.

** Nearly everywhere in the book, we indicate partial derivatives by subscripts.

entropy at the moving particle remains constant, i.e. the changes in state at the particle are *adiabatic*. We shall, therefore, often be interested in $f(\rho, S)$ and $g(\tau, S)$ considered simply as functions of ρ and τ respectively, with the specific entropy S fixed; indeed, in some cases these functions will be written in the abbreviated form $f(\rho)$ and $g(\tau)$. The equation $p = f(\rho) = g(\tau)$ is then called the *adiabatic equation*.

The word *isentropic* would perhaps be more accurate here than *adiabatic*. If, for example, heat conduction were absent but viscosity present, the changes would be adiabatic (heat not flowing to or from the particle), but not isentropic (the entropy at the particle generally increasing). But we are reserving the word *isentropic* for another concept, that of constant entropy throughout the medium.

It is a fundamental property of all actual media that, entropy remaining constant, the pressure increases with increasing density (or decreasing specific volume), that is,

$$f_\rho(\rho, S) > 0; \; g_\tau(\tau, S) < 0, \tag{2.04}$$

except in the limiting case $\rho = 0$, in which $f_\rho = 0$. Because of (2.04) we can define a positive quantity c, with the dimension of speed, by

$$c^2 = \frac{dp}{d\rho} = f_\rho(\rho, S), \qquad \rho^2 c^2 = -g_\tau(\tau, S). \tag{2.05}$$

This important quantity c is called the *sound speed*, a name which will be justified in Section 35, Chapter II; the quantity ρc is frequently called the *acoustic impedance*.

For any value of S, the function $g(\tau, S)$ is generally convex downward. We therefore assume throughout this book, except where the contrary is noted, that

$$g_{\tau\tau}(\tau, S) > 0. \tag{2.06}$$

It is useful to recognize that together with (2.04), $f_{\rho\rho}(\rho, S) \geq 0$ implies (2.06).

We make the additional assumption that, for constant specific volume, the pressure increases with entropy,

$$g_S(\tau, S) > 0. \tag{2.07}$$

From equations (2.02) we see that this assumption is equivalent to

assuming that, at constant entropy, the temperature increases with increasing density.

For gases, for which the density may approach zero, we make the additional assumption

$$e \to 0, \ \tau p \to 0, \ T \to 0, \ c \to 0, \text{ as } \rho \to 0. \tag{2.08}$$

The theory of nonlinear wave motion can be carried quite far without further assumptions about the medium. There are, however, various media of particular physical interest, which are described in Sections 3–6 (in somewhat more detail than is necessary for the subsequent mathematical treatment).

3. *Ideal gases, polytropic gases, and media with separable energy*

In practically all applications of the theory to gases the medium may, with sufficient accuracy, be assumed to be an *ideal gas*, that is a medium which satisfies the laws of Boyle and Gay-Lussac as expressed by the *equation of state*

$$p\tau = RT. \tag{3.01}$$

Here the constant R may be taken to be the universal gas constant R_0 divided by the effective molecular weight of the particular gas.

In an ideal gas the internal energy is a function of the temperature alone, see Section 4. If, in particular, the internal energy is simply proportional to the temperature T, the gas is called *polytropic*. For such gases we may write

$$e = c_v T, \tag{3.02}$$

where the constant c_v is the specific heat at constant volume. The assumption that a gas is polytropic is made in most applications of the theory; it leads, together with (3.01), to the caloric equation of state

$$p = f(\rho, S) = A\rho^\gamma, \tag{3.03}$$

in which the coefficient A depends on the entropy, S and the *adiabatic exponent* γ is a constant between 1 and $\frac{5}{3}$ for media most usually occurring. Air at moderate temperatures may be considered polytropic with $\gamma = 1.4$.

the state of each particle are adiabatic and reversible. The pressure p is here a function of ρ and S. Using

$$\frac{\partial p}{\partial \rho} = c^2, \tag{17.04}$$

see (7.12), equation (17.01) may conveniently be replaced by

$$p_t + up_x + \rho c^2 u_x = 0, \tag{17.05}$$

so that the three equations (17.02), (17.05), and (17.03) involve only the derivatives of u, p, and S. For *polytropic gases*, equation (17.03) may be replaced by

$$(p\rho^{-\gamma})_t + u(p\rho^{-\gamma})_x = 0, \tag{17.06}$$

since $p\rho^{-\gamma}$ is then a function of the entropy. If the flow is *isentropic*, one may express p in terms of ρ or vice versa. Equations (17.02) and (17.01) or (17.05) then represent two equations for two functions of x and t.

Spherical flow occurs when all quantities depend only on the distance from one point, chosen as the origin 0, in addition to the time, and if the velocity is directed away from (or toward) this point. Denoting the distance from the origin conveniently by x and the radial velocity component by u, equations (7.08–.11) reduce to

$$\rho_t + u\rho_x + \rho u_x + 2\rho u/x = 0, \tag{17.07}$$

$$\rho(u_t + uu_x) + p_x = 0, \tag{17.08}$$

$$S_t + uS_x = 0. \tag{17.09}$$

We note that the only difference from the equations for one-dimensional flow is the additional term occurring in the continuity equation (17.07). The modifications and simplifications just discussed in connection with one-dimensional flow apply just as well to spherical flow.

The same remarks apply to *cylindrical flow*, a two-dimensional flow in which all quantities depend only on the distance from the axis and the velocity is directed away from (or toward) the axis. The only difference is that in equation (17.07) the factor 1 occurs instead of the factor 2.

18. Lagrange's equations of motion for one-dimensional and spherical flow

In one-dimensional flow Lagrange's equations are not encumbered by functional determinants, and in this special case they are sometimes preferable to Euler's equations.

The Lagrangian point of view requires us to attach a number h to each plane section of particles normal to the x-axis, so that the changing position of each section is given by a function $x(h, t)$. The quantities ρ, p, S are then considered functions of h and t. This number h could be chosen in many ways; in fact an arbitrary function is at our disposal.

Customarily, one identifies h with the abscissa of the particle at some initial time, e.g. $t = 0$. But not in all problems is such an initial position at a common initial time given for all particles.

Another rather natural choice of h, based on the law of conservation of mass, suggests itself. Without any loss of generality we may think of the flow as taking place in a tube of unit cross section along the x-axis. Now attach the value $h = 0$ to any definite "zero" section (moving, of course, with the medium), and then for any other section let h be equal in magnitude to the mass of the medium in the tube of unit cross section area between that section and the zero section, the sign of h being taken positive or negative according as the zero section is to the left or right of the other section in question. It is clear that h as so defined is different for every section.

Analytically, the quantity h satisfies the relation

$$h = \int_{x(0,t)}^{x(h,t)} \rho \, dx. \tag{18.01}$$

Here ρ is the density at the position x at the time t; in other words, the density is here regarded from the Eulerian point of view as a function of the independent variables x and t. Differentiating (18.01) with respect to h leads to the relation

$$\rho(h, t) x_h(h, t) = 1 \tag{18.02.1}$$

or

$$x_h(h, t) = \tau(h, t), \tag{18.02.2}$$

in which $\rho(h, t)$ and $\tau(h, t)$ are respectively the density and specific volume.

Lagrange's equations of motion (7.02–.05) for one-dimensional flow then take the form:

(18.03) $$(\rho x_h)_t = 0$$

(Conservation of mass),

(18.04) $$\rho x_{tt} = -p_x = -p_h/x_h$$

(Conservation of momentum),

(18.05) $$S_t = 0$$

(Changes of state are adiabatic),

(18.06) $$p = f(\rho, S) = g(\tau, S)$$

(Caloric equation of state).

(We have here abandoned the dot $(\cdot)$ in favor of the subscript t, which will not lead to confusion between Eulerian and Lagrangian concepts.)

Lagrange's equations (18.03–.06) may be simplified considerably. In the first place it follows from (18.02) that (18.03) is superfluous and that (18.04) may be replaced by

(18.07) $$x_{tt} = -p_h.$$

According to (18.05), S depends only on h; we will henceforth always imply $S = S(h)$. The function $S(h)$ is considered as given among the initial conditions of the problem. By means of (18.02) and (18.04–.06) we can eliminate ρ, τ, and p and the whole system reduces to a single partial differential equation of the second order in x:

(18.08) $$x_{tt} = k^2 x_{hh} - g_S S_h,$$

in which we have introduced the quantity

(18.09) $$k = \rho c = \sqrt{-g_\tau(\tau, S)},$$

the *acoustic impedance* of the medium. In interpreting equation (18.08) it must of course be remembered that k^2 and g_S are given functions of $\tau = x_h$ and $S(h)$.

If the velocity $u = x_t$ and the specific volume $\tau = x_h$ are taken as

the dependent variables the single second order equation (18.08) is replaced by the first order system for u and τ

$$u_h = \tau_t \qquad (18.10)$$
$$u_t = k^2 \tau_h - g_S S_h(h),$$

in which k^2 and g_S are functions of τ and $S(h)$.

If the flow is isentropic, i.e. if $S(h) = S_0$ is a constant, equations (18.08) and (18.10) simplify to

$$x_{tt} = k^2 x_{hh} \qquad (18.11)$$

$$u_h = \tau_t \qquad (18.12)$$
$$u_t = k^2 \tau_h .$$

Note that if $k^2 = -p_\tau$ is constant, as it is for a solid which obeys Hooke's law, equations (18.11) and (18.12) are linear.

The formulations of this section can easily be extended to *spherical* and *cylindrical* flow, see Section 17. Let $4\pi h$ denote the mass of the medium inside a sphere of radius $y(h, t)$ about the center of a spherical flow. Then we have

$$y_{tt} = y^2[k^2(y^2 y_h)_h - g_S S_h],$$

analogous to (18.08). For a plane flow with cylindrical symmetry we have in corresponding notation

$$y_{tt} = y^2[k^2(y y_h)_h - g_S S_h].$$

APPENDIX

Wave Motion in Shallow Water

19. Shallow water theory

An analogue to the nonlinear wave motion of gases is encountered in the motion of water, or any other incompressible fluid, with a free top surface if the height of the top surface above the bottom surface is sufficiently small. One speaks then of "shallow water." More

precisely the condition is that the height of the top surface above the bottom surface is small compared with some characteristic length of the motion such as the maximum radius of curvature occurring on the top surface. The differential equations governing the motion of such shallow water can in good approximation be replaced by equations which are completely equivalent to those for a polytropic gas with the exponent $\gamma = 2$. As a matter of fact all the phenomena of wave motion which we shall discuss in subsequent chapters have their strict analogue in the wave motion of shallow water.

We place an (x, y, z)-coordinate system in the space filled by the water in such a way that the bottom surface is the plane $z = 0$ and the top surface is given by a function $z = Z(x, y, t)$. We denote the components of velocity in the x-, y-, and z-directions by u, v, w respectively; u, v, w are functions of x, y, z, t. In the water the continuity condition

(19.01) $$u_x + v_y + w_z = 0$$

and Newton's law

(19.02) $$\rho \frac{du}{dt} = -p_x, \qquad \rho \frac{dv}{dt} = -p_y, \qquad \rho \frac{dw}{dt} = -p_z - g\rho$$

holds. Here g is the acceleration of gravity, ρ the density of the water, and p the excess pressure above atmospheric pressure so that

(19.03) $$p = 0 \text{ at the top surface, } z = Z.$$

The boundary conditions for the velocity are

(19.04) $$w = 0 \text{ at the bottom, } z = 0,$$

and

(19.05) $$Z_t + uZ_x + vZ_y = w \text{ at the top, } z = Z.$$

It is now possible to replace these equations, in good approximation, by equations involving only the top surface elevation Z and the velocities u and v at the top surface. In order to do so we first integrate the continuity equation (19.01) from $z = 0$ to $z = Z$, obtaining

$$W \Big|_{z=0}^{z=Z} + \int_0^Z (u_x + v_y)\, dz = 0$$

from which, by the boundary conditions (19.04) and (19.05),

$$Z_t + \left(\int_0^Z u\,dz\right)_x + \left(\int_0^Z v\,dz\right)_y = 0. \tag{19.06}$$

Next we introduce the *basic assumption that the variation of the pressure along a vertical column is the same as in hydrostatics*

$$p = g\rho(Z - z). \tag{19.07}$$

By (19.02) and (19.03) this assumption is equivalent to the condition that the vertical acceleration of the water, $\frac{dw}{dt}$, vanishes. Assumption (19.07) is not arbitrary; it can be shown to be correct in first order as a consequence of the basic assumption of shallowness by a systematic expansion of the equations in powers of the height z; see Stoker [27, Appendix].

We note that assumption (19.07) implies that the pressure gradient (p_x, p_y, p_z) is independent of z; by (19.02) the same is also true for the acceleration $\left(\frac{du}{dt}, \frac{dv}{dt}, \frac{dw}{dt}\right)$. As a consequence, the velocity (u, v, w) is also independent of z if this was ever the case at some time. We now introduce the assumption, that at some time the velocity was constant over every vertical column. This assumption does not seem to be a serious restriction; it would, for example, be satisfied if the water was at rest at some time. As a consequence u and v depend on x, y, t only; we shall ignore the vertical velocity w from now on.

The equations (19.06) and (19.02) now assume the simple form

$$Z_t + (Zu)_x + (Zv)_y = 0 \tag{19.08}$$

$$\begin{aligned} \rho(u_t + uu_x + vu_y) &= -g\rho Z_x, \\ \rho(v_t + uv_x + vv_y) &= -g\rho Z_y. \end{aligned} \tag{19.09}$$

To show that these equations are equivalent to those for polytropic gases with $\gamma = 2$ we introduce the "density"

$$\bar{\rho} = \rho Z, \tag{19.10}$$

which is evidently the mass per unit area, and the "pressure"

$$\bar{p} = \tfrac{1}{2}g\rho Z^2 = \int_0^Z p\,dz. \tag{19.11}$$

Equations (19.08–.09) can then be written in the form of the equations for gases, see (7.08), (7.09),

$$\bar{\rho}_t + (\bar{\rho}u)_x + (\bar{\rho}v)_y = 0, \tag{19.12}$$

$$\begin{aligned} \bar{\rho}(u_t + uu_x + vu_y) &= -\bar{p}_x, \\ \bar{\rho}(v_t + uv_x + vv_y) &= -\bar{p}_y. \end{aligned} \tag{19.13}$$

The relationship

$$\bar{p} = \frac{g}{2\rho}\,\bar{\rho}^2 \tag{19.14}$$

between "pressure" $\bar{p}$ and "density" $\bar{\rho}$, which follows from (19.10) and (19.11), evidently corresponds to the relationship between the real pressure and density for a polytropic gas with $\gamma = 2$.

CHAPTER II

Mathematical Theory of Hyperbolic Flow Equations for Functions of Two Variables

In the preceding part we have shown that for many specific cases the flow differential equations reduce to *systems of quasi-linear partial differential equations of the first order for functions of two independent variables*. For such systems a fairly complete mathematical theory can be developed provided they are of the *hyperbolic* type, in which case the notion of *characteristics* plays the dominant role. This theory becomes particularly simple when the number of functions and equations is two.

To prepare a deeper understanding of the treatment of specific flow problems in the following chapters we insert here a detailed theory of systems of two differential equations; supplementary remarks will be added about systems of more than two differential equations, as they occur for example in non-isentropic flow.

20. *Flow equations involving two functions of two variables*

For convenient reference we enumerate the specific types of flow which are governed by a system of two equations for two functions of two variables.

a) One-dimensional isentropic flow

$$\rho_t + u\rho_x + \rho u_x = 0, \qquad \rho(u_t + uu_x) + c^2\rho_x = 0 \tag{20.01}$$

see (17.01), (17.02), (17.04).

b) Spherical isentropic flow

$$\rho_t + u\rho_x + \rho u_x + 2\rho u/x = 0, \qquad \rho(u_t + uu_x) + c^2\rho_x = 0 \tag{20.02}$$

see (17.04), (17.07), (17.08).

c) One-dimensional isentropic flow in Lagrangian representation

$$(20.03)\qquad \begin{aligned} \tau_t &= u_h \\ u_t &= k^2 \tau_h , \end{aligned}$$

see (18.12); here k is a given function of τ.

d) One-dimensional non-isentropic flow in Lagrangian representation

$$(20.04)\qquad \begin{aligned} \tau_t &= u_h \\ u_t &= k^2 \tau_h - g_S S_h , \end{aligned}$$

see (18.10); here the distribution of the entropy $S = S(h)$ over the particles is assumed to be given, k^2 and g_S are given functions of τ and S.

e) Steady two-dimensional irrotational isentropic flow

$$(20.05)\qquad \begin{aligned} v_x - u_y &= 0 \\ (c^2 - u^2)u_x - uv(u_y + v_x) + (c^2 - v^2)v_y &= 0, \end{aligned}$$

see (16.06) and (16.08); here c^2 is a given function of $u^2 + v^2$.

f) Steady irrotational isentropic flow in three dimensions with cylindrical symmetry

$$(20.06)\qquad v_x - u_y = 0$$

$$(c^2 - u^2)u_x - uv(u_y + v_x) + (c^2 - v^2)v_y + c^2 v/y = 0,$$

see (16.06) and (16.14); again c^2 is a given function of $u^2 + v^2$.

21. Differential equations of second order type

In the general theory we denote by u, v the dependent and by x, y the independent variables, to be identified later with the variables in the specific differential equations of gas dynamics which we have enumerated in the preceding section. Then the general form of the system of differential equations is

$$(21.01)\qquad \begin{aligned} L_1 &= A_1 u_x + B_1 u_y + C_1 v_x + D_1 v_y + E_1 = 0 \\ L_2 &= A_2 u_x + B_2 u_y + C_2 v_x + D_2 v_y + E_2 = 0 \end{aligned}$$

in which $A_1, A_2, \cdots, E_2$ are known functions of x, y, u, v. Once